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ON MINIMAL, STRONGLY PROXIMAL ACTIONS OF LOCALLY COMPACT GROUPS

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ABSTRACT

Minimal, strongly proximal actions of locally compact groups on compact spaces, also known as *boundary* actions, were introduced by Furstenberg in the study of Lie groups. In particular, the action of a semi-simple real Lie group G on homogeneous spaces G/Q, where $Q \subset G$ is a parabolic subgroup, are boundary actions. Countable discrete groups admit a wide variety of boundary actions. In this note we show that if X is a compact manifold with a faithful boundary action of some locally compact group H, then (under some mild regularity assumption) the group H, the space X, and the action split into a direct product of a semi-simple Lie group G acting on G/Q and a boundary action of a discrete countable group.

1. Introduction

Let G be a locally compact group (hereafter *locally compact* groups are always assumed to be second countable). A compact Hausdorff space X with a jointly continuous G-action $G \times X \to X$, $(g, x) \mapsto g \cdot x$, will be called a G-**space.** A G-space X is **minimal** (or the G-action on X is minimal) if X has no proper closed G-invariant subsets; equivalently if *every* G-orbit $G \cdot x$ is dense in X. By Zorn's lemma every compact G-space X contains a closed G-invariant set $X' \subseteq X$ which is a minimal G-space.

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Given a compact G-space X consider the set $\mathcal{P}(X)$ of all Borel probability measures on X equipped with the weak-* topology induced by continuous functions $\mathcal{C}(X)$. Then $\mathcal{P}(X)$ is a convex compact subset of the unit ball in $\mathcal{C}(X)^*$, equipped with a **continuous affine action** of G induced by the G-action on X. In dynamics one often studies G-spaces X which support an invariant probability measure, i.e., G-spaces for which the affine G-action on $\mathcal{P}(X)$ has a fixed point. Amenable groups are precisely characterized by the property that every G-space admits G-invariant probability measures. In this note we shall be interested in G-spaces X which exhibit an opposite behavior. More precisely

Definition 1 (Furstenberg, [2])*:* A compact G-space X is called **strongly proximal** if for every probability measure $\mu \in \mathcal{P}(X)$ the G-orbit $G\cdot\mu \subset \mathcal{P}(X)$ contains Dirac measures δ_x in its weak-* closure. A G-space which is minimal and strongly proximal will be called a G-**boundary** (we shall also say that the G-action on X is a **boundary action**).

By definition X is a G-boundary iff every orbit $G \cdot \mu$ in the affine G-action on $\mathcal{P}(X)$ contains the set $\delta_X = \{\delta_x \mid x \in X\}$ in its closure. Since δ_X is the set $\mathrm{Ext}\mathcal{P}(X)$ of extremal points of $\mathcal{P}(X)$, one has

(1) A G-space X is a G-boundary iff the affine G-action on $\mathcal{P}(X)$ admits no proper closed convex invariant subsets.

One can also consider the following more general setup: let E be a locally convex topological vector space with a continuous affine G-action, and $V \subset E$ be a convex compact G-invariant subset. The restriction of the affine G-action to V will be called an **affine representation**. An affine G-representation on V is **irreducible** if V does not contain G-invariant closed convex proper subsets. The following basic facts summarize some of the results and observations made by Furstenberg who introduced the above notions of boundaries and affine representations [2], [3] (see also Glasner's [5]):

(2) An affine G-representation on $V \subset E$ is irreducible iff the G-action on the closure $\overline{\mathrm{Ext}(V)}$ of the set of all extremal points of V is a boundary action.

(3) Any quotient of a G-boundary is a G-boundary, i.e., if p: $X \to Y$ is a continuous surjection between G-spaces X and Y and X is a G-boundary then so is Y.

(4) Given a locally compact group G there exists a unique, up to isomorphisms, **maximal** G-**boundary** $B(G)$ which is *universal* in the sense that any G-boundary X can be obtained as a quotient p_X: $B(G) \to X$ with the continuous G-equivariant surjection p_X being uniquely determined. Similarly, there exists a unique maximal irreducible affine G-representation

$V(G)$ which is universal in the sense that for any irreducible affine G-representation W there exists a unique continuous G-equivariant affine surjection $q_W: V(G) \to W$. It follows that $V(G) = \mathcal{P}(B(G))$. The universal boundary $B(G)$ is a Hausdorff compact space, but in general it need not be metrizable (resp. $V(G)$ need not be separable).

(5) A locally compact group G is *amenable* iff $B(G)$ is trivial, i.e., is a point. More generally, if G is a locally compact group containing a closed amenable subgroup P so that G/P is compact, then the universal G-boundary $B(G)$ is a G-equivariant image of G/P by a unique continuous surjective map $\pi: G/P \to B(G)$, and every G-boundary X is obtained as a G-equivariant quotient G/Q where $Q \supseteq P$ is a closed subgroup. In particular, if G is a connected semi-simple real Lie group with finite center and no compact factors, then $B(G) = G/P$ with $P = AN$ being a minimal parabolic, where $G = KAN$ is the Iwasawa decomposition of G. In this case $B(G)$ is also known as the **Furstenberg boundary**.

(6) Let $\Gamma \subset G$ be a lattice in a locally compact group G. Then the Γ-action on $B(G)$ is a boundary action. However, typically $B(G)$ is not the maximal Γ-boundary.

The purpose of this note is to describe boundary actions of general locally compact groups G and to discuss to what extent the structure of a G-boundary X (or $B(G)$) as a *topological space* determines the group G.

2. Statement of the results

Let X be a compact n-manifold without a boundary, let d be a metric on X associated with some smooth Riemannian structure and assume that H is a locally compact group with a continuous action by homeomorphisms on X. We shall say that this action satisfies the following property

(Hr) H acts by $\frac{n}{n+2}$-Hölder homeomorphisms, or more precisely that there exists a neighborhood U of the identity in H so that for each $h \in U$ there is a constant $C = C(h)$ so that

$$C^{-1} \cdot d(x,y)^{\frac{n+2}{n}} \leq d(h \cdot x, h \cdot y) \leq C \cdot d(x,y)^{\frac{n}{n+2}}.$$

(QC) H acts by quasi-conformal maps on X with respect to the conformal structure defined by a smooth Riemannian metric on X.

THEOREM 2: *Let X be a compact n-manifold which is an H-boundary where H is a locally compact group and let $H_1 := H/\operatorname{Ker}(H \to \operatorname{Homeo}(X))$. Assume that either* (C) *$n = 1$, i.e., X is a circle, or that the H_1-action on X satisfies* (Hr), *or the H_1-action on X satisfies* (QC). *Then H_1 contains a subgroup H_2 of finite index such that*

- *The H_2-action on X is a boundary action.*
- *The group H_2 is isomorphic to a direct product $G \times \Lambda$, where Λ is either trivial or is an infinite discrete countable group, G is either trivial or is a connected real semi-simple Lie group with trivial center and no non-trivial compact factors; the index $[H_1 : H_2]$ divides $|\operatorname{Out}(G)|$.*
- *The manifold X is homeomorphic to a direct product of compact spaces $Y \times Z$, where Y is G-boundary (isomorphic to $Y \cong G/Q$) and Z is an Λ-boundary, so that the $H_2 \cong G \times \Lambda$-action on $X \cong Y \times Z$ corresponds to the G-action on $Y = G/Q$ and the Λ-action on Z.*

COROLLARY 3: *Let H be a locally compact group with a faithful boundary action on a compact manifold X which does not split as a non-trivial direct product of topological spaces. If the H-action on X satisfies one of the assumptions* (C), (Hr), (QC) *above, then*

- *either H is a discrete infinite countable group, or*
- *H is a connected semi-simple real Lie group G with trivial center and no non-trivial compact factors, and as a G-space X is isomorphic to G/Q where $Q \subsetneq G$ is a parabolic subgroup.*

The assumptions (C), (Hr), (QC) in the theorem are used to ensure that the locally compact group H_1 satisfies **No Small Subgroups** (abbreviated, **NSS**) property, which means that H_1 has a neighborhood U of the identity that does not contain non-trivial subgroups; or equivalently, that H_1 contains a neighborhood U' of the identity which contains no non-trivial *compact* subgroups. Real Lie groups and discrete groups are typical examples of groups with NSS property; while p-adic Lie groups and other totally disconnected non-discrete groups, such as the group of automorphism of a regular tree, have families of "small subgroups". However, for groups acting faithfully on topological manifolds the following generalization of Hilbert's 5-th problem is conjectured to hold:

HILBERT–SMITH CONJECTURE: *Every locally compact group acting faithfully by homeomorphisms an a topological manifold has NSS property.*

This conjecture is known to hold in the following cases:

(a) If $\dim X = 1$, i.e., X is a topological circle $S^1 = \mathbb{R}/\mathbb{Z}$. This follows from the fact that any compact subgroup of $\operatorname{Homeo}_+(S^1)$ is conjugate to a subgroup of the rotation group SO(2), which has NSS property.

(b) Let X be an n-manifold equipped with some Riemannian metric d. If $H \subset \mathrm{Homeo}(X)$ is a locally compact group, such that some neighborhood U of the identity in H consists of $\frac{n}{n+2}$-Hölder homeomorphisms, then H has NSS property (Maleshich [7]).

(c) Let X be a differentiable manifold with a conformal structure on it and H be a locally compact group acting by quasi-conformal maps on X; then H has NSS property (Martin [8]).

Assumptions (C), (Hr) and (QC) correspond to (a), (b) and (c) above. If/when proved, the Hilbert-Smith conjecture will allow us to drop these assumptions in the theorem.

Remarks 4:

(i) Observe that for the circle $X = S^1$ the theorem asserts that any locally compact group H acting faithfully, minimally and strongly proximally on S^1 is either discrete or is isomorphic to either $\mathrm{PSL}_2(\mathbb{R})$ or to its double cover $\mathrm{PGL}_2(\mathbb{R})$, so that the action is continuously conjugate to the standard projective action of these groups on $\mathbb{RP}^1 \approx S^1$. In fact, in the case of the circle it can be shown that the only non-discrete locally compact groups with faithful minimal (but not necessarily strongly proximal) actions on S^1 are $\mathrm{PSL}_2(\mathbb{R})$, $\mathrm{PGL}_2(\mathbb{R})$, $\mathrm{SO}(2)$ and $O(2)$. However, the class of discrete groups with a faithful boundary action on the circle is already huge and includes, besides many discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$, Thompson groups and fundamental groups of many 3-manifolds (see Ghys [4]).

(ii) If one is willing to assume the Hilbert–Smith conjecture (or to apply conditions (Hr) or (QC), when relevant) then one could deduce from Corollary 3 that

- The only non-discrete locally compact groups with a faithful boundary action on the sphere $X = S^2$ are $G = \mathrm{Isom}_+(\mathbb{H}^3)$ and its double cover $\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{Aut}(G)$ (these are the simple Lie group $G = \mathrm{PO}(3,1) \cong PSL_2(\mathbb{C})$ and its group of automorphisms) with the action being conjugate to the standard one, when one identifies the Riemann sphere S^2 with the boundary $\partial\mathbb{H}^3$ of the hyperbolic 3-space $\mathbb{H}^3$.
- The only non-discrete locally compact groups with a faithful boundary action on the projective plane $X = \mathbb{RP}^2$ is $\mathrm{SL}_3(\mathbb{R})$ with the action being conjugate to the standard (i.e., projective) one.
- There are no non-discrete locally compact groups with a faithful boundary action on a compact surface Σ_g of genus $g \geq 2$.

(iii) For spheres $X = S^k$ one obtains several families of non-discrete locally compact groups with a faithful boundary action on S^k, namely G and its double cover $\mathrm{Aut}\, G$ where G is the following rank-one simple Lie group: $G = \mathrm{PO}(n,1)$ with $k = n-1$, $G = \mathrm{PU}(n,1)$ with $k = 2n-1$, $G = \mathrm{PSp}(n,1)$ with $k = 4n-1$, $G = \mathrm{F}_{4(-20)}$ with $k = 15$.

(iv) At the same time one can show that any manifold admits a variety of boundary actions by a free group F_2, so one should not expect to classify boundary actions of discrete groups even on nice spaces.

The question of describing non-discrete locally compact groups with a faithful boundary action on a manifold appeared in the context of the following

PROBLEM: *Given a discrete finitely generated group* Γ, *describe all locally compact groups* H *which admit a* **cocompact lattice embedding** *of* Γ, *i.e., an embedding* $j\colon \Gamma \to H$ *with* $j(\Gamma)$ *being a discrete subgroup in* H *with* $H/j(\Gamma)$ *compact.*

Observe that any discrete group Γ forms a cocompact lattice in semi-direct products $\Gamma \ltimes K$ by any compact group K (and any homomorphism $\Gamma \to \mathrm{Aut}\, K$) via the embedding $\gamma \mapsto (\gamma, e_K)$. These constructions, and their obvious modifications obtained by passing to finite index subgroups, are "trivial" examples of cocompact lattice embeddings. A potentially non-trivial class of examples of cocompact lattice embeddings can be obtained by considering the Cayley graph $X_{\Gamma,\Sigma}$ of Γ with respect to some finite symmetric set Σ of generators and taking H to be $\mathrm{Aut}(X_{\Gamma,\Sigma})$ — the totally disconnected group of all automorphisms of the Cayley graph $X_{\Gamma,\Sigma}$. In addition, fundamental groups $\Gamma = \pi_1 M$ of compact locally symmetric manifolds M have natural cocompact lattice embeddings in the semi-simple Lie group $G = \mathrm{Isom}_+(\tilde{M})$, and in direct products $H = G \times K$ where K is an arbitrary compact group.

In [1] cocompact lattice embeddings of Γ in an arbitrary locally compact group H were classified for fundamental groups $\Gamma = \pi_1 M$ of compact (and finite volume, in higher rank cases) locally symmetric manifolds M, i.e., groups Γ which admit lattice embeddings in a semi-simple Lie group G. In these cases it was proven that, up to finite index and centers, the natural Γ-embeddings in semi-direct products $\Gamma \ltimes K$ and $G \times K$ (in both cases K is compact) are the only examples of cocompact lattice embeddings. In particular, for these groups one always has $[\mathrm{Aut}(X_{\Gamma,\Sigma}) : \Gamma] < \infty$.

Recall that Gromov and Thurston [6] proved that in each dimension $n \geq 4$ there exist compact manifolds M which admit Riemannian structures of strictly negative curvature, but do not carry a locally symmetric structure. Very little is known about the structure of the fundamental groups $\Gamma = \pi_1 M$ of such manifolds, beyond the fact that these are Gromov hyperbolic groups which do not embed as cocompact lattices in semi-simple Lie groups. However, assuming the Hilbert–Smith conjecture one would be able to deduce the following:

COROLLARY 5: *Let $\Gamma = \pi_1 M$ be the fundamental group of a compact manifold M which admits a Riemannian structure of strictly negative curvature, but does not admit a locally symmetric one. Let H be a locally compact group which admits an embedding $j\colon \Gamma \to H$ with $j(\Gamma)$ being a cocompact lattice in H. Then the Hilbert Smith conjecture implies that Γ is contained in a closed subgroup $H_0 \subseteq H$ of finite index in H, so that H_0 is isomorphic to a semi-direct product $\Gamma \ltimes K$ where K is a compact group. In particular, for any finite symmetric set Σ of generators for Γ the Cayley graph $X_{\Gamma,\Sigma}$ admits at most finite number of automorphisms, up to translations by Γ, i.e., $[\mathrm{Aut}(X_{\Gamma,\Sigma}) : \Gamma] < \infty$.*

Remark 6: Consider $\Gamma = \pi_1 M$ as above and let $H := \mathrm{Isom}(\tilde{M})$ where the universal cover $\tilde{M}$ of M is equipped with the lift of some Riemannian metric on M. In this case it is well known that H is a locally compact group with NSS property, and therefore H is a discrete group containing Γ as a finite index subgroup. More generally, if (M, g) is a negatively curved manifold, then $\mathrm{Isom}(\tilde{M})$ is either discrete or $\tilde{M}$ is a symmetric space in which case $\mathrm{Isom}(\tilde{M})$ is a rank-one simple Lie group.

This note is organized as follows. The proof of Theorem 2 is divided into two steps: splitting the acting group (section 4) and splitting the space and the actions (section 5). Application 5 is derived in section 6. We preface the discussion with a general remark about the *amenable radical.*

3. The amenable radical

Recall that a locally compact group H is amenable iff every compact H-space X has an H-invariant probability measure or, equivalently, every affine H-representation has a fixed point. Moreover, for this characterization it suffices to consider only compact *metric* H-spaces (i.e., separable affine H-representation). We shall use this opportunity to generalize this fact.

PROPOSITION 7 (The Amenable Radical): *Let H be a locally compact group. Consider the following subgroups of H:*

(1) $N_0 = \mathrm{Ker}(H \to \mathrm{Homeo}(B(H)))$.

(2) $N_1 = \bigcap_{i \in I} \mathrm{Ker}(H \xrightarrow{\rho_i} \mathrm{Homeo}(X_i))$, *where* $\{H \to \mathrm{Homeo}(X_i)\}_{i \in I}$ *is the collection of all isomorphism classes of boundary* H*-actions on compact metric spaces* X_i.

(3) N_{am} — *the group generated by all closed normal amenable subgroups in* H.

Then $N_0 = N_1 = N_{am}$ *is the maximal closed normal amenable subgroup of* H*, which can be called the* **amenable radical** *of* H.

Proof: Let N be a closed normal amenable subgroup of H and X be a compact metric H-boundary. Denote by $\mathcal{P}_N \subseteq \mathcal{P}(X)$ the set of all N-invariant probability measures on X. This is a closed convex subset of the compact convex set $\mathcal{P}(X)$. Since N is amenable, $\mathcal{P}_N$ is non-empty; and since N is normal, $\mathcal{P}_N$ is an invariant set for the affine H-action on $\mathcal{P}(X)$. As the latter affine H-action is irreducible we have $\mathcal{P}_N = \mathcal{P}(X)$, and in particular $\delta_X \subset \mathcal{P}_N$. Hence N acts trivially on X. This argument applies to all compact (metric) boundary H-actions X, which means that $N \subseteq N_1 = \bigcap_{i \in I} \mathrm{Ker}(H \to \mathrm{Homeo}(X_i))$. Hence $N_{am} \subseteq N_1$.

By the maximality of $B(H)$ we have $N_1 \subseteq N_0$ and the latter is a closed normal subgroup of H. Therefore, proving that N_0 is amenable would imply $N_0 \subseteq N_{am}$ and yield the equalities $N_0 = N_1 = N_{am}$.

Assume that N_0 is not amenable. Then one can find a continuous N_0-action on some compact metric space M which has no invariant measures. A standard construction of induction allows one to induce linear or affine representations from a closed subgroup (here N_0) to the larger group H (see Zimmer [10] 4.2 for details). In our case, consider the space $L^\infty(H/N_0, \mathcal{C}(M)^*) = L^1(H/N_0, \mathcal{C}(M))^*$ with the weak-* topology, and its convex compact subset W consisting of all classes of Borel functions μ: $H/N_0 \to \mathcal{P}(M)$, μ: $x \mapsto \mu_x \in \mathcal{P}(M)$, where μ is identified with μ' if $\mu_x = \mu'_x$ for a.e. $x \in H/N_0$. Choosing a measurable cross section $\sigma : H/N_0 \to H$ of the projection π: $H \to H/N_0$, one can define a measurable cocycle α: $H \times H/N_0 \to N_0$ by

$$\alpha(h, h'N_0) = \sigma(hh'N_0)^{-1} h \sigma(h'N_0)$$

and verify that the H-action on W defined by

$$(h \cdot \mu)_x = \alpha(h, x) \cdot \mu_x, \quad x \in H/N_0$$

gives a continuous (!) affine representation of H. Let $V \subseteq W$ be a *minimal* H-invariant convex compact set (i.e., an irreducible affine H-representation). Since N_0 acts trivially on $B(H)$, it acts trivially in the universal irreducible affine H-representation, and thereby trivially on V. Observe that since N_0 is normal in

H, for every $h \in N_0$ and almost every $x \in H/N_0$

$$h \cdot x = x \quad \text{and} \quad \alpha(h, x) = \sigma(x)^{-1} h \sigma(x). \tag{3.1}$$

Fix an μ: $H/N_0 \to \mathcal{P}(M)$ from V. Then for a.e. $x \in H/N_0$ the measure $\mu_x \in \mathcal{P}(M)$ is fixed by $\sigma(x)^{-1} h \sigma(x)$ for a.e. $h \in N_0$ (Fubini theorem applied to (3.1)), and therefore by the whole N_0. This contradicts the assumption. Hence N_0 is amenable and the proof is completed. ∎

We shall use the following immediate corollary (explicitly shown at the beginning of the proof):

COROLLARY 8: *A closed normal amenable subgroup N of a locally compact group H acts trivially on every H-boundary X.*

4. Splitting the group with NSS property

PROPOSITION 9: *Let H be a locally compact group with a faithful boundary action on a compact space X. Assume that H has NSS property. Then H contains a closed normal subgroup H_0 of finite index in H, which still acts minimally and strongly proximally on X and is isomorphic to a direct product $H_0 \cong G \times \Lambda$, where G is either trivial or is a connected semisimple real Lie group with trivial center and no non-trivial compact factors, and Λ is either trivial or is an infinite discrete countable group. The index $[H : H_0]$ divides $|\operatorname{Out}(G)|$.*

Proof: Denote by G the connected component of the identity in H. By the fundamental results of Montgomery and Zippin ([9]), the assumption that H has NSS property means that G is a connected Lie group. It is normal in H and the factor group $\Lambda := H/G$ is a totally disconnected locally compact group.

Assume that G is non-trivial. Observe that any closed amenable characteristic subgroup of G is a closed amenable normal subgroup in H and therefore is trivial by Corollary 8. Hence G has trivial radical, trivial center and no compact factors.

Next observe that H acts on G by conjugation, which gives rise to a homomorphism

$$H \longrightarrow \operatorname{Aut} G.$$

Recall that $\operatorname{Aut} G$ contains $\operatorname{Ad} G \cong G$ as a finite index subgroup. Let H_0 denote the preimage of G and let $\Lambda = Z_{H_0}(G)$ denote the centralizer of G in H_0. By the definition of H_0 for each $h \in H_0$ there is a (unique) $g_h \in G$, so that

$$h^{-1} g h = g_h^{-1} g g_h \quad (g \in G),$$

which means that $H_0 = G \cdot \Lambda$, and moreover H_0 is (isomorphic to) the direct product $G \times \Lambda$. Since Λ is isomorphic to H_0 modulo its connected component of the identity G, the group Λ is totally disconnected. At the same time being a closed subgroup of the NSS group H_0, Λ has to be *discrete*.

Finally, observe that a restriction of the boundary H-action on X to a finite index subgroup $H_0 \subseteq H$ is still a boundary action. (This is the simplest case of fact (6) from the introduction.) Indeed, let $h_1 = e, h_2, \ldots, h_n$ be some representatives of the cosets H/H_0, and let V be a minimal closed convex H_0-invariant subset of $\mathcal{P}(X)$. Consider the collection $W \subseteq \mathcal{P}(X)$ of all probability measures of the form

$$\mu = \frac{h_1\mu_1 + \cdots + h_n\mu_n}{n} \quad \text{where } \mu_i \in V.$$

Then W is a closed convex H-invariant subset of $\mathcal{P}(X)$ and is therefore $W = \mathcal{P}(X)$. In particular $\delta_X \subseteq W$. Since δ_X consists of extremal points of $\mathcal{P}(X)$ we have $\delta_X \subseteq V$ and therefore $V = \mathcal{P}(X)$. Thus X is an H_0-boundary. ∎

5. Splitting the space and the action

THEOREM 10: *Let X be an H-boundary for a locally compact group H which is a direct product $H = G \times L$ of two locally compact groups. Assume that G can be written as $G = K \cdot P$, where $K \subseteq G$ is a compact subgroup and $P \subseteq G$ is a closed amenable subgroup. Then there is a homeomorphism $\theta\colon X \to Y \times Z$, $\theta(x) = (\phi(x), \psi(x))$, identifying X with a direct product of a G-boundary Y and an L-boundary Z, so that*

$$\theta((g,l)\cdot x) = (g \cdot \phi(x), l \cdot \psi(x)).$$

The G-space Y can be identified with G/Q where Q is a closed subgroup $P \subseteq Q \subseteq G$.

The assumption $G = K \cdot P$ (for a semi-simple Lie group G this is the Iwasawa decomposition) is used in the following key Lemma.

LEMMA 11: *$K \subset G$ acts transitively on G-orbits. In particular, every G-orbit $G \cdot x \subset X$ is compact.*

Proof: Denote by $\mathcal{P}_P \subseteq \mathcal{P}(X)$ the set of all P-invariant probability measures on X. This is a *non-empty* convex compact subset of $\mathcal{P}(X)$, which is L-invariant because L commutes with P. Observe that the set

$$G \cdot \mathcal{P}_P = \{k \cdot \mu \in \mathcal{P}(X) \mid k \in K, \mu \in \mathcal{P}_P\}$$

is a non-empty *closed* subset of $\mathcal{P}(X)$, which is still L-invariant by commutativity. For every $\mu \in \mathcal{P}_P$ the H-orbit $H \cdot \mu$ satisfies

$$H \cdot \mu = (G \times L) \cdot \mu = L \cdot (G \cdot \mu) \subset L(G \cdot \mathcal{P}_P) = G \cdot \mathcal{P}_P.$$

By definition of boundary actions every Dirac measure δ_x is contained in a closure of $H \cdot \mu \subset G \cdot \mathcal{P}_P$, and since the set $G \cdot \mathcal{P}_P$ is already *closed*, we conclude that $\delta_X \subseteq G \cdot \mathcal{P}_P$. Therefore, for every $x \in X$ the G-orbit $G \cdot \delta_x$ intersects $\mathcal{P}_P$, i.e., every G-orbit $G \cdot x$ in X contains a P-fixed point. Since $G = K \cdot P$ the group K acts transitively on each G-orbit. ∎

Every G-orbit $G \cdot x$ in X can therefore be identified with G/G_x, where $G_x := \{g \in G \mid g \cdot x = x\}$ denotes the G-stabilizer of $x \in X$. G_x are closed subgroup of G.

PROPOSITION 12: *All G-stabilizers G_x are conjugate (in G) to each other.*

For the proof of the Proposition we shall need to compare "sizes" of G-orbits. Since K acts transitively on G-orbits (Lemma 11) every G-orbit $G \cdot x \cong G/G_x$ can be viewed as K/K_x where $K_x = \{k \in K \mid k \cdot x = x\}$. Consider a partial order $\preceq$ between conjugacy classes $[K']$ of closed subgroups $K' \subseteq K$, with $[K'] \preceq [K'']$ if there exists $k \in K$ so that $K' \subseteq k^{-1}K''k$. (Thus the orbit $G \cdot x$ is as "large" as $G \cdot y$ if $[K_x] \preceq [K_y]$.) Observe that

$$[K'] \preceq [K''] \quad \text{and} \quad [K''] \preceq [K'] \quad \text{implies} \quad [K'] = [K''].$$

To see this it suffices to check that if K' is a closed subgroup of K and $k \in K$ satisfies $k^{-1}K'k \subseteq K'$, then $k^{-1}K'k = K'$. This is evident for finite groups and for compact connected Lie groups, and hence follows for all compact groups which are inverse limits of the former families of compact groups.

LEMMA 13: *If $x_n \to x_*$ in X and $[K_{x_n}] \preceq [K_{x_{n+1}}]$ for all n, then $[K_{x_n}] \preceq [K_{x_*}]$ for all n.*

Proof: Replacing, if necessary, x_n's by $y_n = k_n \cdot x_n$ with an appropriate $k_n \in K$, we can assume that $K_{y_n} \subseteq K_{y_{n+1}}$ for all n, and passing to a convergent subsequence we may assume that $y_n \to y_* = k \cdot x_*$ with $k \in K$. Then $K_{y_n} \subseteq K_{y_*}$ for all n in the subsequence, because for $k \in K_{y_n}$,

$$k \cdot y_* = k \cdot \lim_{n\to\infty} y_n = \lim_{n\to\infty} k \cdot y_n = \lim_{n\to\infty} y_n = y_*,$$

so that $k \in K_{y_*}$, and therefore $[K_{x_n}] = [K_{y_n}] \preceq [K_{y_*}] = [K_{x_*}]$. Since this applies to any subsequence of the original sequence the lemma is proved. ∎

Proof Proposition 12: Since any sequence in X contains a convergent subsequence, Lemma 13 shows that there exist points $x \in X$ with *maximal stabilizers*, i.e., points x with the property that if $[K_x] \preceq [K_y]$ then $[K_y] = [K_x]$. Let x_* be such a point with a maximal stabilizer K_{x_*} and let $X_* = \{x \in X \mid [K_x] = [K_{x_*}]\}$. Then X_* is a non-empty *closed* set. Indeed, if $x_n \to y$ and $x_n \in X_*$, then by the last lemma $[K_{x_*}] = [K_{x_n}] \preceq [K_y]$ and by the maximality of $[K_{x_*}]$ we have $[K_{x_*}] = [K_y]$, i.e., $y \in X_*$. Hence X_* is closed. The set X_* is clearly G-invariant and also L-invariant, by commutativity. We conclude that $X_* = X$ because $H = G \times L$ acts minimally on X. Thus all stabilizers K_x are mutually conjugate, and therefore all G-stabilizers G_x, $x \in X$, are conjugate in G. ∎

Observe that in the case of a semi-simple Lie group G the proof of Proposition 12 can be simplified, since stabilizers G_x in G are parabolic subgroups and there are only finitely many conjugacy classes of those in G, and the partial order argument is not needed.

Now let us fix a point $o \in X$ and denote $Y := G/G_o$. Proposition 12 implies that for every $x \in X$ there exists $g_x \in G$, so that $G_x = g_x^{-1} G_0 g_x$.

LEMMA 14: *The map $\phi\colon X \to Y = G/G_o$ given by $\phi(x) = g_x G_o$ is a well defined continuous map.*

Proof: Assume that for some $x \in X$ both g_1 and g_2 in G satisfy

$$G_x = g_1^{-1} G_o g_1 = g_2^{-1} G_o g_2.$$

Then $(g_1 g_2^{-1})^{-1} G_o (g_1 g_2^{-1}) = G_o$, so that $g_2 g_1^{-1} \cdot o = o$ and therefore $g_1 G_o = g_2 G_o \in G/G_o$. Hence $\phi\colon X \to G/G_o$ is indeed well defined.

To verify continuity of ϕ, consider $x_n \to x$ in X. Since G/G_o is compact, upon passing to a subsequence and replacing g_{x_n} by g_{y_n} with $g_{x_n} G_o = g_{y_n} G_o$, we can assume that $g_{y_n} \to g_y$ in G. Thus $g_{y_n} \cdot x_n \to g_y \cdot x$ while $g_{y_n}^{-1} G_o g_{y_n} \cdot x_n = x_n$. This means that every $g \in G_o$ satisfies

$$g \cdot (g_{y_n} \cdot x_n) = g_{y_n} \cdot x_n, \quad g \cdot (g_y \cdot x) = g_y \cdot x,$$

which shows that $g_y^{-1} G_o g_y = G_x$, i.e., $\phi(y) = g_y G_o = g_x G_o = \phi(x)$, proving $\phi(x_n) \to \phi(x)$. ∎

LEMMA 15: *For any L-minimal set $Z \subset X$ and any G-orbit $G \cdot x$ the intersection $Z \cap G \cdot x$ consists of at most one point.*

Proof: As G and L commute, $G_x = G_{l \cdot x}$ for all $l \in L$ and $x \in X$. Hence $\phi(x)$ is constant on L-orbits. Since ϕ is continuous (Lemma 14) it has to be constant on any minimal L-set. On the other hand, on every G-orbit $G \cdot x = G/G_x$ the map $\phi: G/G_x \to G/G_o$ is a bijection (actually homeomorphism). Hence $|Z \cap G \cdot x| \leq 1$. ∎

By Lemma 11 all G-orbits on X are compact and therefore the projection $\pi: X \to X/G$ is a continuous surjective map between compact spaces, which is equivariant with respect to the H-action on X and the well defined L-action on X/G. Hence X/G is an L-boundary.

Fix an L-minimal set $Z \subseteq X$. The projection $\pi(Z) \subseteq X/G$ is a non-empty closed L-invariant set and therefore $\pi(Z) = X/G$ by the minimality of the L-action on X/G. Lemma 15 states that $\pi: Z \to X/G$ is one-to-one and therefore is an L-equivariant homeomorphism. Thus for every $x \in X$ there is a unique point $z_x \in Z$ so that $\pi(x) = \pi(z_x)$, and therefore there is a $g \in G$ so that $x = g \cdot z_x$. The map $Z \to gZ$ given by $z \mapsto g \cdot z$ is an L-equivariant homeomorphism, so that gZ is also an L-minimal set, and the whole space X is a disjoint union of L-minimal sets gZ.

One can now replace the minimal L-set Z by an appropriate g-translate, so that the reference point $o \in X$ would be in Z. After this adjustment define the map $\psi: X \to Z$ by $\pi(x) = \pi(\psi(x))$, and observe that for all $x \in X$, $g \in G$, $l \in L$ one has

$$\psi((g,l) \cdot x) = l \cdot \psi(x), \quad \phi((g,l) \cdot x) = g \cdot \phi(x).$$

Since the maps $\psi: X \to Z$ and $\phi: X \to G/G_o$ are continuous, so is

$$\theta: X \to Y \times Z, \quad \theta(x) = (\phi(x), \psi(x)).$$

Finally, by Lemma 15 the map θ is one-to-one. This completes the proof of Theorem 10. ∎

Theorem 2 now follows from the cited results which guarantee the NSS property for the faithfully acting group, Proposition 9 and Theorem 10.

6. Proof of Corollary 5

Let $\Gamma = \pi_1 M$ be a fundamental group of a compact n-manifold M which admits a negatively curved Riemannian structure, but does not carry a locally symmetric one, and let $j: \Gamma \to H$ be a cocompact lattice embedding of Γ in a locally compact group H. Denote by $\partial\Gamma$ the (ideal) boundary of the hyperbolic group Γ, which can be identified with the (visual) boundary $\partial\tilde{M}$ of the universal cover $\tilde{M}$ of M and is, therefore, homeomorphic to the sphere S^{n-1}. The natural continuous Γ-action on $\partial\Gamma \cong S^{n-1}$, which we shall denote by $\rho: \Gamma \to \mathrm{Homeo}(\partial\Gamma)$, is faithful minimal and strongly proximal (in fact Γ acts as a convergence group on its boundary).

It is shown in [1] (Theorem 3.5) that given an embedding $j\colon \Gamma \to H$ as a cocompact lattice, there exists a homomorphism

$$\Psi = \Psi_j\colon H \longrightarrow \mathrm{Homeo}(\partial\Gamma)$$

so that $\Psi \circ j\colon \Gamma \to \mathrm{Homeo}(\partial\Gamma)$ coincides with $\rho\colon \Gamma \to \mathrm{Homeo}(\partial\Gamma)$ and, with respect to the uniform topology on $\mathrm{Homeo}(\partial\Gamma)$, the homomorphism Ψ is continuous, has a compact kernel $K_1 = \mathrm{Ker}(\Psi)$ and a closed image $H_1 = \Psi(H) \subset \mathrm{Homeo}(\partial\Gamma)$, so that H_1 is a locally compact group which contains $\rho(\Gamma) \cong \Gamma$ as a cocompact lattice. Therefore H_1 acts as a convergence group on $\partial\Gamma \cong S^{n-1}$, and in particular S^{n-1} is an H_1-boundary.

Assuming the Hilbert–Smith conjecture, H_1 satisfies the NSS property. Since S^{n-1} does not split as a non-trivial direct product of spaces, by Theorem 2 either H_1 contains a finite index subgroup H_2 isomorphic to a semi-simple connected Lie group G, or H_2 is discrete. (One could also argue that Gromov hyperbolic groups, such as Γ, cannot be embedded as a cocompact lattice in a direct product of two non-compact groups, so for $H_2 = G \times \Lambda$ either Λ or G is trivial.) The first possibility, H_1 being a semi-simple Lie group, is excluded by the assumption that M does not carry any locally symmetric Riemannian structure. Hence H_1 is a discrete countable group, which contains $\rho(\Gamma)$ as a cocompact lattice, i.e., as a finite index subgroup. Let $H_0 = \Psi^{-1}(\rho(\Gamma))$ and $K = K_1 \cap H_0$. We have an exact sequence

$$1 \longrightarrow K \longrightarrow H_0 \longrightarrow \rho(\Gamma) \longrightarrow 1$$

which splits by $\rho(\Gamma) \xrightarrow{\rho^{-1}} \Gamma \xrightarrow{j} H_0$. Hence H_0 is isomorphic to a semi-direct product $\Gamma \ltimes K$, where Γ acts on K by $\gamma\colon k \to j(\gamma)^{-1} k j(\gamma)$.

Now let Σ be a finite symmetric generating set for Γ and let $X_{\Gamma,\Sigma}$ denote the corresponding Cayley graph. $H := \mathrm{Aut}(X_{\Gamma,\Sigma})$ is a locally compact group containing Γ (acting by translations) as a cocompact lattice. Indeed, H/Γ can be identified with a stabilizer K_γ of a vertex v_γ in $X_{\Gamma,\Sigma}$ which is a compact group. Hence $\Gamma \subseteq H_0 \subseteq H$ with $[H : H_0] < \infty$ and $H_0 \cong \Gamma \ltimes K$ for some compact normal group $K \subset H_0$. We claim that K has to be *finite*, so that after passing to the kernel H_1 of $H_0 \to \mathrm{Aut}\, K$ which still has a finite index in $H = \mathrm{Aut}(X_{\Gamma,\Sigma})$, one would obtain $H_1 \subseteq \Gamma$. To see that K is finite, observe that for $k \in K$ one has

$$k(v_\gamma) = k(\gamma(v_e)) = \gamma k^\gamma(v_e),$$

where $k^\gamma = \gamma^{-1}k\gamma \in K$. As K is compact, the orbit $\{k(v_e) \mid k \in K\}$ is bounded, which means that

$$d(k(v_\gamma), v_\gamma) \leq D < \infty$$

for some fixed D and all $v_\gamma \in V := V(X_{\Gamma,\Sigma})$, $k \in K$. Balls $B(v, D) := \{u \in V \mid d(v, u) \leq D\}$ have at most $N := |\Sigma|^D$ elements. Since every $k \in K$ together with all its powers belongs to K, it permutes vertices in each ball $B(v, D)$, and therefore $k^{N!}$ fixes every v. Hence $k^{N!} = e$ for all $k \in K$, i.e., K is finite. The corollary is proved. ∎

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